

A RADIATION CONDITION FOR UNIQUENESS IN A WAVE PROPAGATION PROBLEM FOR 2-D OPEN WAVEGUIDES

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Abstract. We study the uniqueness of solutions of Helmholtz equation for a problem that concerns wave propagation in waveguides. The classical radiation condition does not apply to our problem because the inhomogeneity of the index of refraction extends to infinity in one direction. Also, because of the presence of a waveguide, some waves propagate in one direction with different propagation constants and without decaying in amplitude.

Our main result provides an explicit condition for uniqueness which takes into account the physically significant components, corresponding to guided and non-guided waves; this condition reduces to the classical Sommerfeld-Rellich condition in the relevant cases.

Finally, we also show that our condition is satisfied by a solution, already present in literature, of the problem under consideration.

Key words. Electromagnetic fields, Wave propagation, Helmholtz equation, optical waveguides, uniqueness of solutions, radiation condition.

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1. The problem of uniqueness for the Helmholtz equation. Let $\Sigma \subset \mathbb{R}^N$ be a (possibly empty) bounded closed surface. It is well known that the Dirichlet problem

$$\begin{cases} \Delta u + k^2 u = f & \text{outside } \Sigma, \\ u = U & \text{on } \Sigma, \end{cases} \quad (1.1)$$

has not an unique solution. If $k = 0$ (Poisson's equation), in order to obtain the uniqueness, it is required that the solution vanishes at infinity. If $k \neq 0$, that is not sufficient anymore. In fact, there are two different solutions of (1.1) which vanish at infinity, representing the outward and inward radiation. Hence, an additional (or different) condition at infinity is needed.

The first condition we can add is that

$$\lim_{R \rightarrow \infty} R^{\frac{N-1}{2}} \left(\frac{\partial u}{\partial R} - iku \right) = 0, \quad (1.2)$$

uniformly; this is the so-called *Sommerfeld's radiation condition*. Here, $\frac{\partial u}{\partial R}$ denotes the radial derivative of u . The physical meaning of this condition is that there are no sources of energy at infinity. Moreover, it assures that, far from the surface Σ , u behaves as a wave generated by a point source.

Stated as in (1.2) and together with the assumption that u vanishes at infinity, this condition is due to Sommerfeld, see [So1] and [So2] (see also [Mag1] and [Mag2]). The vanishing assumption on u was dropped by Rellich (see [Rel]), who also proved that (1.2) can be replaced by the weaker condition

$$\lim_{R \rightarrow \infty} \int_{\partial B_R} \left| \frac{\partial u}{\partial R} - iku \right|^2 d\sigma = 0, \quad (1.3)$$

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where B_R is the ball centered at the origin with radius R . In the same paper, Rellich also proved that a radiation condition can also be given in the form

$$\int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial R} - iku \right|^2 dx < +\infty. \quad (1.4)$$

Condition (1.4) can be considered the starting point for our work, as we are going to explain shortly. Before describing our results, we cite some generalizations of the work of Rellich.

When n is a function which is identically 1 outside a compact set, (1.3) still guarantees the uniqueness of a solution of

$$\Delta u + k^2 n(x)^2 u = f, \quad x \in \mathbb{R}^N;$$

see [Mi1] and [Sc] and references therein.

Several authors (see [Rel] [Mi2] [RS] [Zh] [PV] [Ei]) studied the case in which n is not constant at infinity, but has an angular dependency, say $n(x) \rightarrow n_\infty(x/|x|)$ as $|x| \rightarrow \infty$, and it approaches to the limit with a certain behaviour. Among these papers, we want to mention the results in [Zh] and [PV], where the authors proved the uniqueness of solutions of the Helmholtz equation by means of the limiting absorption method and by introducing the radiation condition:

$$\lim_{R \rightarrow +\infty} \frac{1}{R} \int_{B_R} \left| \frac{\partial u}{\partial R} - ikn_\infty u \right|^2 dx = 0.$$

Here, the assumptions on n are so that the energy cannot be trapped along any direction, but it radiates toward infinity. That is in accordance with [PS], where the authors point out that the Sommerfeld radiation condition, since it involves the dimension, is inappropriate for problems admitting a lower dimensional solution (a plane wave).

The present paper is motivated by the study of wave propagation in optical waveguides. In particular, we shall study the uniqueness of solutions of the two-dimensional Helmholtz equation

$$\Delta u + k^2 n(x)^2 u = f, \quad (x, z) \in \mathbb{R}^2, \quad (1.5)$$

where n is of the form

$$n := \begin{cases} n_{co}(x), & |x| \leq h, \\ n_{cl}, & |x| > h; \end{cases} \quad (1.6)$$

here n_{co} is a bounded function and n_{cl} is a constant; thus, (1.6) models the index of refraction of a rectilinear open waveguide of width $2h$ (subscripts *co* and *cl* refer to the *core* and *cladding* of the waveguide) (see [SL]).

We observe that functions n of type (1.6) are not considered in the works cited before. In fact, the most important feature of optical waveguides is the presence of waves confined inside the waveguide (*guided modes*) which are oscillatory and never decaying along the axis of propagation (z -axis). It is easy to show that a pure guided mode supported by the Helmholtz equation does not satisfy the radiation conditions above retrieved (as already pointed out in [PS]). Functions n similar to (1.6) were

considered by Jäger and Saitō in [JS1]-[JS2]; however, their assumptions on n do not admit the occurrence of guided modes.

As far as we know, the only works dealing with uniqueness in an optical waveguide setting have appeared in the Russian literature (see [Rei] [No] [NS] [KNH] and references therein). However, the *Reichardt condition* studied therein only deals with guided modes and does not apply to the total field.

The main result of this paper is Theorem 2.3, where we present a new radiation condition that guarantees the uniqueness of a solution of (1.5) with n given by (1.6). We observe that, if we suppose that no guided mode is present (this is possible by choosing special parameters in the function n), our radiation condition reduces to (1.4). In this setting, our results provide a different proof of special cases studied in [Rel] and [JS1]-[JS2].

The key ingredients of our proof are essentially four: (i) if (1.5) possesses two solutions satisfying our radiation condition, then their difference w must belong to the Sobolev space $H^2(\mathbb{R}^2)$; (ii) as a consequence of (i), the Fourier transform of w in the z -direction (parallel to the fiber's axis) is square integrable for almost all $x \in \mathbb{R}$ and satisfies an ordinary differential equation in x ; (iii) the only square integrable solution of such an equation is identically zero; (iv) the proof is then completed by using an appropriate transform theory in the x -direction and repeating the arguments in (ii) and (iii). This scheme will be carried out in §2.

In [MS] the authors derived a solution¹ for the problem (1.5)-(1.6) in terms of a Green's function. Section 3 is devoted to prove that such a solution satisfies our radiation condition. This will be done in three steps: in §3.1 we derive a representation of the solution as a contour integral; in §3.2 we prove uniform estimates for the non-guided part of the *spectrum-based solution*; in §3.3 we carry out the proof by testing the radiation condition on the guided part and using the asymptotic estimates obtained in §3.2.

We wish to observe that the results in the present paper can be easily adapted to prove the uniqueness of a solution for the Pekeris waveguide problem (see [Wi] and [De]).

2. A new Rellich-type condition and a uniqueness theorem. In this section we shall state a radiation condition, that generalizes (1.4), and prove our uniqueness result.

2.1. Preliminaries. We recall the relevant results of [MS] which will be useful in the rest of the paper.

In [MS] a Green's function G for (1.5) is constructed: a solution of (1.5) is

$$u(x, z) = \int_{\mathbb{R}^2} G(x, z; \xi, \zeta) f(\xi, \zeta) d\xi d\zeta, \quad (2.1)$$

where

$$G(x, z; \xi, \zeta) = \sum_{j \in \{s, a\}} \int_0^{+\infty} \frac{e^{i|z-\zeta|\sqrt{k^2 n_*^2 - \lambda}}}{2i\sqrt{k^2 n_*^2 - \lambda}} v_j(x, \lambda) v_j(\xi, \lambda) d\rho_j(\lambda). \quad (2.2)$$

Here

$$n_* = \max_{\mathbb{R}} n,$$

¹We will refer to such a solution as the *spectrum-based solution*.

$$\langle d\rho_j, \eta \rangle = \sum_{m=1}^{M_j} r_m^j \eta(\lambda_m^j) + \frac{1}{2\pi} \int_{d^2}^{+\infty} \frac{\sqrt{\lambda - d^2}}{(\lambda - d^2)\phi_j(h, \lambda)^2 + \phi_j'(h, \lambda)^2} \eta(\lambda) d\lambda,$$

for all $\eta \in C_0^\infty(\mathbb{R})$ ($\langle \cdot, \cdot \rangle$ is the usual dual product),

$$r_m^j = \left[\int_{-\infty}^{+\infty} v_j(x, \lambda_m^j)^2 dx \right]^{-1} = \frac{\sqrt{d^2 - \lambda_m^j}}{\sqrt{d^2 - \lambda_m^j} \int_{-h}^h \phi_j(x, \lambda_m^j)^2 dx + \phi_j(h, \lambda_m^j)^2}. \quad (2.3)$$

Also, $v_j(x, \lambda)$ are linearly independent solutions of

$$v'' + [\lambda - q(x)]v = 0, \quad \text{in } \mathbb{R}, \quad (2.4)$$

where $q(x) = k^2[n_*^2 - n(x)^2]$, and have the following form:

$$v_j(x, \lambda) = \begin{cases} \phi_j(h, \lambda) \cos Q(x - h) + \frac{\phi_j'(h, \lambda)}{Q} \sin Q(x - h), & \text{if } x > h, \\ \phi_j(x, \lambda), & \text{if } |x| \leq h, \\ \phi_j(-h, \lambda) \cos Q(x + h) + \frac{\phi_j'(-h, \lambda)}{Q} \sin Q(x + h), & \text{if } x < -h, \end{cases} \quad (2.5)$$

for $j = s, a$, with $Q = \sqrt{\lambda - d^2}$, $d^2 = k^2(n_*^2 - n_{cl}^2)$; the ϕ_j 's are solutions of (2.4) in the interval $(-h, h)$ and satisfy the initial conditions:

$$\begin{aligned} \phi_s(0, \lambda) &= 1, & \phi_s'(0, \lambda) &= 0, \\ \phi_a(0, \lambda) &= 0, & \phi_a'(0, \lambda) &= \sqrt{\lambda}. \end{aligned}$$

(The indices $j = s, a$ correspond to symmetric and antisymmetric solutions, respectively.)

We notice that (2.2) can be split up into two summands,

$$G = G^g + G^{rad},$$

where

$$G^g(x, z; \xi, \zeta) = \sum_{j \in \{s, a\}} \sum_{m=1}^{M_j} \frac{e^{i|z-\zeta|\sqrt{k^2 n_*^2 - \lambda_m^j}}}{2i\sqrt{k^2 n_*^2 - \lambda_m^j}} v_j(x, \lambda_m^j) v_j(\xi, \lambda_m^j) r_m^j, \quad (2.6)$$

and

$$G^{rad}(x, z; \xi, \zeta) = \frac{1}{2\pi} \sum_{j \in \{s, a\}} \int_{d^2}^{+\infty} \frac{e^{i|z-\zeta|\sqrt{k^2 n_*^2 - \lambda}}}{2i\sqrt{k^2 n_*^2 - \lambda}} v_j(x, \lambda) v_j(\xi, \lambda) \frac{\sigma_j(\lambda)}{\sqrt{\lambda - d^2}} d\lambda, \quad (2.7)$$

with

$$\sigma_j(\lambda) = \frac{\lambda - d^2}{(\lambda - d^2)\phi_j(h, \lambda)^2 + \phi_j'(h, \lambda)^2}, \quad j = s, a; \quad (2.8)$$

G^g represents the guided part of the Green's function, which involves the guided modes, i.e. the modes propagating mainly inside the waveguide; G^{rad} is the part of

the Green's function corresponding to the non-guided energy, i.e. the energy radiated outside or vanishing along the waveguide, which we denote by

$$u^{rad} = \int_{\mathbb{R}^2} G^{rad}(x, z; \xi, \zeta) f(\xi, \zeta). \quad (2.9)$$

It exists a finite number of guided modes, which corresponds to the finite number of roots of the equations

$$\sqrt{d^2 - \lambda} \phi_j(h, \lambda) + \phi'_j(h, \lambda) = 0, \quad j \in \{s, a\},$$

laying in the interval $(0, d^2)$. We shall denote by λ_m^j , $m = 1, \dots, M_j$, $j = s, a$, these roots. Each $v_j(x, \lambda_m^j)$ decays exponentially for $|x| > h$ as it is clear from the formula:

$$v_j(x, \lambda_m^j) = \begin{cases} \phi_j(h, \lambda_m^j) e^{-\sqrt{d^2 - \lambda_m^j}(x-h)}, & x > h, \\ \phi_j(x, \lambda_m^j), & |x| \leq h, \\ \phi_j(-h, \lambda_m^j) e^{\sqrt{d^2 - \lambda_m^j}(x+h)}, & x < -h. \end{cases} \quad (2.10)$$

We notice that G^g is bounded and oscillatory in the z direction, because $\sqrt{k^2 n_*^2 - \lambda_m^j}$ is real for every $m = 1, \dots, M_j$, $j = s, a$.

REMARK 2.1. *The functions $\sigma_j(\lambda)$, $j \in \{s, a\}$, given by (2.8), are meromorphic functions of $\lambda \in \mathbb{C}$, real-valued for $\lambda \in \mathbb{R}$ and with poles that are real and simple (see [CL], [Ti]), which corresponds to the values λ_m^j , $m = 1, \dots, M_j$, $j = s, a$.*

To simplify notations, we shall denote by γ_l , $l = 1, \dots, M$, $M = M_s + M_a$, the values λ_m^j , $m = 1, \dots, M_j$, $j = s, a$, ordered according to the natural ordering on the real line, and by γ_* their maximum. With these premises, we set

$$e(x, \gamma_l) = \frac{v_j(x, \gamma_l)}{\|v_j(\cdot, \gamma_l)\|_2}. \quad (2.11)$$

From (2.6) and (2.3), it is clear that the guided part G^g can be written as

$$G^g(x, z; \xi, \zeta) = \sum_{l=1}^M G_l^g(x, z; \xi, \zeta),$$

where

$$G_l^g(x, z; \xi, \zeta) = \frac{e^{i\beta_l|z-\zeta|}}{2i\beta_l} e(x, \gamma_l) e(\xi, \gamma_l), \quad (2.12)$$

with

$$\beta_l = \sqrt{k^2 n_*^2 - \gamma_l}. \quad (2.13)$$

Let $s \in \mathbb{R}$; we will denote by $L^{2,s}(\mathbb{R}^2)$ the weighted Lebesgue space consisting of all the complex-valued measurable functions u such that $(1 + x^2 + z^2)^s |u(x, z)|^2$ is summable in \mathbb{R}^2 , equipped with the natural norm defined by

$$|u|_{2,s}^2 = \int_{\mathbb{R}^2} |u(x, z)|^2 (1 + x^2 + z^2)^s dx dz;$$

$L^{2,s}(\mathbb{R}^2)$ is commonly used when dealing with solutions of Helmholtz equation (see [Ag] and [Le]). In [CM] we proved that the spectrum-based solution (2.1)-(2.2) derived in [MS] belongs to $L^{2,s}(\mathbb{R}^2)$, for $s < -1$, if $f \in L^{2,-s}(\mathbb{R}^2)$.

The following lemma will be useful in the next subsection.

LEMMA 2.2. *For $s < -1$, let $w \in L^{2,s}(\mathbb{R}^2)$ satisfy*

$$\Delta u + k^2 n(x)^2 u = 0 \quad (2.14)$$

in \mathbb{R}^2 , where n is given by (1.6). Then

$$\lim_{|x| \rightarrow +\infty} u(x, z) e^{-|x| \sqrt{d^2 - \gamma_*}} = \lim_{|x| \rightarrow +\infty} u_x(x, z) e^{-|x| \sqrt{d^2 - \gamma_*}} = 0, \quad (2.15)$$

where $\gamma_* = \max_{1 \leq l \leq M} \gamma_l$.

Proof. Since u is a solution of (2.14), from Lemmas A.1 and A.3 in [CM], we infer that both $(1 + x^2 + z^2)^s |\nabla u(x, z)|^2$ and $(1 + x^2 + z^2)^s |\nabla^2 u(x, z)|^2$ are summable in \mathbb{R}^2 . Thus, it easily follows that the function

$$\Psi(x, z) = (1 + x^2 + z^2)^{s/2} u(x, z)$$

belongs to the Sobolev space $W^{2,2}(\mathbb{R}^2)$. The Sobolev Imbedding Theorem (see Theorem 4.12 in [AF]) implies that $\Psi \in L^\infty(\mathbb{R}^2)$ and hence the first limit in (2.15) follows at once.

A straightforward computation shows that Ψ satisfies the following equation

$$\Delta \Psi + b \cdot \nabla \Psi + c \Psi = 0$$

in \mathbb{R}^2 , where

$$b(x, z) = -\frac{2s(x, z)}{1 + x^2 + z^2}, \quad c(x, z) = k^2 n(x)^2 - s \frac{2 - s(x^2 + z^2)}{(1 + x^2 + z^2)^2}.$$

Since $\Psi \in W^{2,2}(\mathbb{R}^2)$, by Theorem 8.10 in [GT], we have that $\Psi \in W^{3,2}(H_+)$ where $H_+ = \{(x, z) \in \mathbb{R}^2 : x \geq h\}$. Again, by the Sobolev Imbedding Theorem, $|\nabla \Psi|$ is bounded in H_+ and hence the second limit in (2.15) holds as $x \rightarrow +\infty$. The same limit as $x \rightarrow -\infty$ holds by a similar argument. \square

2.2. The radiation condition and uniqueness theorem. We consider a solution u of (1.5) and define

$$u_l(x, z) = e(x, \gamma_l) U(z, \gamma_l), \quad l = 1, \dots, M, \quad (2.16)$$

with $e(x, \gamma_l)$ given by (2.11) and where

$$U(z, \gamma_l) = \int_{-\infty}^{\infty} u(\xi, z) e(\xi, \gamma_l) d\xi, \quad l = 1, \dots, M. \quad (2.17)$$

The remainder part of u is

$$u_0(x, z) = u(x, z) - \sum_{l=1}^M u_l(x, z). \quad (2.18)$$

We introduce a one-parameter family of sets Ω_R , $R > 0$, such that $\partial\Omega_R$ are level sets of a convex and coercive function $d(x, z)$, i.e. $\Omega_R = \{(x, z) \in \mathbb{R}^2 : d(x, z) \leq R\}$.

With these notations, we state our *radiation condition* for a solution u of (1.5):

$$\sum_{l=0}^M \int_0^\infty \int_{\partial\Omega_R} \left| \frac{\partial u_l}{\partial \nu} - i\beta_l u_l \right|^2 d\ell dR < +\infty, \quad (2.19)$$

with $\beta_0 = kn_{cl}$ and $\beta_l, l = 1, \dots, M$, given by (2.13).

Notice that, when $n \equiv 1$, we can choose $\Omega_R = B_R$ and (2.19) reduces to (1.4), since in such a case the guided components are not present.

The main result of this paper follows.

THEOREM 2.3. *There is at most one solution of (1.5) that satisfies (2.19) and belongs to $u \in L^{2,s}(\mathbb{R}^2)$, $s < -1$.*

REMARK 2.4. *As it will be clear, it is not necessary to specify further the sets Ω_R in (2.19) to get uniqueness of a solution of (1.5). That means that Theorem 2.3 holds for any choice of one-parameter family of sets Ω_R satisfying the above mentioned assumptions.*

Of course, a solution of (1.5) satisfying (2.19) may not exist for an arbitrary choice of the sets Ω_R . In §3, we shall choose a special family of sets Ω_R and prove that, with this choice, the solution of (1.5) given by (2.1)-(2.2) satisfies (2.19).

We also notice that it is not necessary to choose the same sets Ω_R in each addendum in (2.19); Theorem 2.3 still holds if we replace (2.19) by the more general radiation condition

$$\sum_{l=0}^M \int_0^\infty \int_{\partial\Omega_R^{(l)}} \left| \frac{\partial u_l}{\partial \nu} - i\beta_l u_l \right|^2 d\ell dR < +\infty, \quad (2.20)$$

where $\Omega_R^{(l)}$, $l = 0, 1, \dots, M$, are one-parameter families satisfying the above mentioned assumptions.

Theorem 2.3 is based on Lemma 2.5 and Theorem 2.6 below.

LEMMA 2.5. *Let $\beta \in \mathbb{R}$ and u be a weak solution of (2.14) Then*

$$\int_{\partial\Omega} \left| \frac{\partial w}{\partial \nu} - i\beta w \right|^2 d\ell = \int_{\partial\Omega} \left(\left| \frac{\partial w}{\partial \nu} \right|^2 + \beta^2 |w|^2 \right) d\ell, \quad (2.21)$$

for every $\Omega \subset \mathbb{R}^2$ bounded and sufficiently smooth.

Proof. Since u is a weak solution of (2.14), by Theorem 8.8 in [GT], we obtain the necessary regularity to infer that

$$\begin{aligned} \int_{\partial\Omega} \bar{u} \frac{\partial u}{\partial \nu} d\ell &= \int_{\Omega} \operatorname{div}(\bar{u} \nabla u) dx dz = \int_{\Omega} \{ |\nabla u|^2 + \bar{u} \Delta u \} dx dz = \\ &= \int_{\Omega} \{ |\nabla u|^2 - k^2 n(x)^2 |u|^2 \} dx dz. \end{aligned}$$

Therefore

$$\operatorname{Im} \int_{\partial\Omega} \bar{w} \frac{\partial w}{\partial \nu} d\ell = 0,$$

which easily implies (2.21). \square

THEOREM 2.6. *Let $u \in L^{2,s}(\mathbb{R}^2)$ be a weak solution of (2.14) satisfying (2.19). Then*

$$\sum_{l=0}^M \int_0^{+\infty} dR \int_{\partial\Omega_R} \left[\left| \frac{\partial u_l}{\partial \nu} \right|^2 + \beta_l^2 |u_l|^2 \right] d\ell < +\infty, \quad (2.22)$$

and, in particular,

$$\int_{\mathbb{R}^2} |u_l|^2 dx dz < +\infty, \quad (2.23)$$

for every $l = 0, 1, \dots, M$.

Proof. By Lemma 2.5, it is enough to prove that each u_l , $l = 0, 1, \dots, M$, satisfies (2.14). Then, (2.22) and (2.23) will follow from (2.21) and (2.19).

Suppose $l \geq 1$. Since

$$-\int_{\mathbb{R}^2} \nabla u \cdot \nabla \varphi dx dz + k^2 \int_{\mathbb{R}^2} n(x)^2 u \varphi dx dz = 0 \quad (2.24)$$

for every $\varphi \in H_0^1(\mathbb{R}^2)$, we choose $\varphi(x, z) = e(x, \gamma_l) \eta(z)$ with $\eta \in C_0^1(\mathbb{R})$, and obtain:

$$\begin{aligned} & -\int_{\mathbb{R}^2} [u_x(x, z) e'(x, \gamma_l) \eta(z) + u_z(x, z) e(x, \gamma_l) \eta'(z)] dx dz + \\ & k^2 \int_{\mathbb{R}^2} n(x)^2 u(x, z) e(x, \gamma_l) \eta(z) dx dz = 0; \end{aligned}$$

an integration by parts and Lemma 2.2 then give

$$\begin{aligned} & \int_{\mathbb{R}^2} u(x, z) e''(x, \gamma_l) \eta(z) dx dz - \int_{\mathbb{R}^2} u_z(x, z) e(x, \gamma_l) \eta'(z) dx dz + \\ & k^2 \int_{\mathbb{R}^2} n(x)^2 u(x, z) e(x, \gamma_l) \eta(z) dx dz = 0. \end{aligned}$$

Since $e(x, \gamma_l)$ satisfies (2.4), we obtain

$$-\int_{\mathbb{R}^2} u_z(x, z) e(x, \gamma_l) \eta'(z) dx dz + (k^2 n_*^2 - \gamma_l) \int_{\mathbb{R}^2} u(x, z) e(x, \gamma_l) \eta(z) dx dz = 0,$$

and thus, from (2.17),

$$-\int_{\mathbb{R}} U_z(z, \gamma_l) \eta'(z) dz + (k^2 n_*^2 - \gamma_l) \int_{\mathbb{R}} U(z, \gamma_l) \eta(z) dz = 0,$$

for every $\eta \in C_0^1(\mathbb{R})$.

Together with (2.4), this formula implies that each $u_l(x, z)$, $l = 1, \dots, M$, given by (2.16), is a weak solution of (2.14). In fact, for $\varphi(x, z) = \psi(x)\eta(z)$ with $\psi, \eta \in C_0^1(\mathbb{R})$, integration by parts gives

$$\begin{aligned} & - \int_{\mathbb{R}^2} \nabla u_l(x, z) \cdot \nabla \varphi(x, z) \, dx dz + k^2 \int_{\mathbb{R}^2} n(x)^2 u_l(x, z) \varphi(x, z) \, dx dz = \\ & \left(\int_{\mathbb{R}} \{e''(x, \gamma_l) + [\gamma_l - q(x)]e(x, \gamma_l)\} \psi(x) \, dx \right) \left(\int_{\mathbb{R}} U(z, \gamma_l) \eta(z) dz \right) + \\ & \left(\int_{\mathbb{R}} e(x, \gamma_l) \psi(x) dx \right) \left(\int_{\mathbb{R}} [-U_z(z, \gamma_l) \eta'(z) + (k^2 n_*^2 - \gamma_l) U(z, \gamma_l)] \eta(z) dz \right) = 0; \end{aligned}$$

the same conclusion holds for any $\varphi \in C_0^1(\mathbb{R}^2)$, by a density argument.

Since u and u_l , $l = 1, \dots, M$, now satisfy (2.14), the same holds for u_0 . Thus, as already mentioned, we can apply Lemma 2.5 to each u_l , $l = 0, 1, \dots, M$, and obtain

$$\sum_{l=0}^M \int_{\partial\Omega_R} \left| \frac{\partial u_l}{\partial \nu} - i\beta_l u_l \right|^2 d\ell = \sum_{l=0}^M \int_{\partial\Omega_R} \left(\left| \frac{\partial u_l}{\partial \nu} \right|^2 + \beta_l^2 |u_l|^2 \right) d\ell,$$

for every $R > 0$, and then, since u satisfies (2.19), we get (2.22) and (2.23). \square

2.3. Proof of Theorem 2.3. Let u_1 and u_2 be two solutions; $u = u_1 - u_2$ satisfies (2.14) and (2.19).

From Theorem 2.6 we have that $u \in L^2(\mathbb{R}^2)$ and, by using Lemmas A.1 and A.3 in [CM], we get $u \in H^2(\mathbb{R}^2)$. Therefore, $u(x, \cdot) \in L^2(\mathbb{R})$ for almost every $x \in \mathbb{R}$, and the same holds for $u_x(x, \cdot)$ and $u_{xx}(x, \cdot)$. Hence, we can transform (2.14) by using the Fourier transform in the z -coordinate,

$$\hat{u}(x, t) = \int_{-\infty}^{+\infty} u(x, z) e^{-izt} dz, \quad \text{for a.e. } x \in \mathbb{R},$$

and obtain:

$$\hat{u}_{xx}(x, t) + [k^2 n(x)^2 - t^2] \hat{u}(x, t) = 0, \quad \text{a.e. } x \in \mathbb{R}. \quad (2.25)$$

From Fubini-Tonelli's theorem, the integrals

$$\int_{\mathbb{R}^2} |\hat{u}(x, t)|^2 dx dt, \quad \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} |\hat{u}(x, t)|^2 dx \quad \text{and} \quad \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} |\hat{u}(x, t)|^2 dt$$

have the same value, finite or infinite.

Since $u(x, \cdot)$ belongs to $L^2(\mathbb{R})$ for almost every $x \in \mathbb{R}$, the same holds for $\hat{u}(x, \cdot)$ and, furthermore, we have

$$\int_{-\infty}^{+\infty} |\hat{u}(x, t)|^2 dt = 2\pi \int_{-\infty}^{+\infty} |u(x, z)|^2 dz \quad \text{a.e. } x \in \mathbb{R}.$$

By integrating the above equation and using Fubini-Tonelli's theorem, we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} |\hat{u}(x, t)|^2 dx dt &= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} |\hat{u}(x, t)|^2 dt = 2\pi \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} |u(x, z)|^2 dz \\ &= 2\pi \int_{\mathbb{R}^2} |u(x, z)|^2 dx dz < +\infty. \end{aligned}$$

Therefore $\hat{u}(\cdot, t) \in L^2(\mathbb{R})$ for almost every $t \in \mathbb{R}$.

From (2.25), it follows that

$$\hat{u}(x, t) = a(t) \cos \sqrt{\lambda - d^2}(x - h) + b(t) \sin \sqrt{\lambda - d^2}(x - h), \quad \text{for } x > h,$$

where $\lambda = k^2 n_*^2 - t^2$ and $d^2 = k^2(n_*^2 - n_{cl}^2)$. Since

$$\int_{-\infty}^{+\infty} |\hat{u}(x, t)|^2 dx \geq \int_h^{+\infty} |\hat{u}(x, t)|^2 dx,$$

we obtain that $\hat{u}(x, t)$ can be *not identically zero* only for some values $t = \lambda_m^j \in (0, d^2]$ and, furthermore, in that case

$$\hat{u}(x, t) = a(t) v_s(x, \lambda_s^m) + b(t) v_a(x, \lambda_a^m).$$

Hence, for some $A, B \in \mathbb{R}$ we should have

$$u(x, z) = AZ_s(z) v_s(x, \lambda_s^m) + BZ_a(z) v_a(x, \lambda_a^m),$$

where $Z_j(z) = e^{\pm z \sqrt{k^2 n_*^2 - \lambda_m^j}}$, because u is a solution of (2.14). Since $u(x, \cdot) \in L^2(\mathbb{R})$, then both A and B must be zero and hence $u \equiv 0$ on \mathbb{R}^2 .

3. The spectrum-based solution satisfies the radiation condition. It will be useful to introduce the following function

$$[x]_h = \begin{cases} x + h, & x < -h, \\ 0, & -h \leq x \leq h, \\ x - h, & x > h. \end{cases} \quad (3.1)$$

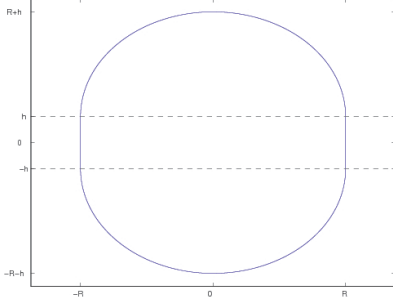
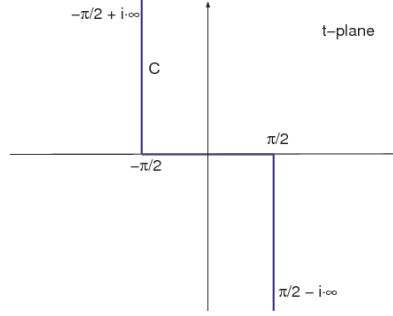
This section is devoted to the proof of the following result.

THEOREM 3.1. *Let $f \in L^2(\mathbb{R}^2)$ be such that $f \equiv 0$ a.e. outside a compact subset of \mathbb{R}^2 . Then, the spectrum-based solution (2.1) of (1.5) is the only solution in $L^{2,s}(\mathbb{R}^2)$, $s < -1$, such that*

$$\int_0^\infty \int_{\partial \Omega_R} \left| \frac{\partial u_0}{\partial \nu} - i \beta_0 u_0 \right|^2 d\ell dR + \sum_{l=1}^M \int_0^\infty \int_{\partial Q_R} \left| \frac{\partial u_l}{\partial \nu} - i \beta_l u_l \right|^2 d\ell dR < +\infty, \quad (3.2)$$

where Ω_R is given by

$$\Omega_R = \{(x, z) \in \mathbb{R}^2 : [x]_h^2 + z^2 \leq R^2\} \quad (3.3)$$

FIGURE 3.1. The set Ω_R .FIGURE 3.2. The contour \mathcal{C} .

(see Fig. 3.1) and $Q_R = \{(x, z) \in \mathbb{R}^2 : |x|, |z| \leq R\}$.

REMARK 3.2. At the cost of extra computations, it may be proved that Theorem 3.1 also holds if we replace (2.20) by the more compact condition (2.19) with Ω_R given by (3.3).

We shall break the proof of Theorem 3.1 up into three steps. First, in §3.1, we will derive a handier representation of the radiating part G^{rad} of the Green's function, as a suitable contour integral (see Lemma 3.3). Then, in §3.2, we will prove a uniform asymptotic expansion for the quantity $\frac{\partial G^{rad}}{\partial \nu} - i\beta_0 G^{rad}$ on the sets $\partial\Omega_R$. Such an expansion will be used in §3.3 to carry out the proof of Theorem 3.1, where we also test the radiation condition on the guided components of u .

3.1. Representing G^{rad} as a contour integral. We introduce the following functions:

$$\{x\}_h = x - [x]_h,$$

with $[x]_h$ given by (3.1), and, for $\tau \in \mathbb{C}$,

$$\Phi_j(x, \tau) = \phi_j(\{x\}_h, d^2 + \tau^2) + \frac{\phi'_j(\{x\}_h, d^2 + \tau^2)}{i\tau}, \quad j \in \{s, a\}. \quad (3.4)$$

With these notations, (2.5) and (2.8) take the more compact forms:

$$v_j(x, d^2 + \tau^2) = \frac{1}{2} \left\{ \Phi_j(x, \tau) e^{i\tau[x]_h} + \Phi_j(x, -\tau) e^{-i\tau[x]_h} \right\} \quad (3.5)$$

and

$$\sigma_j(d^2 + \tau^2) = \frac{1}{\Phi_j(h, \tau) \Phi_j(h, -\tau)}, \quad (3.6)$$

for $j \in \{s, a\}$.

LEMMA 3.3. Let \mathcal{C} be the contour from $-\frac{\pi}{2} + i \cdot \infty$ to $\frac{\pi}{2} - i \cdot \infty$ shown in Fig. 3.2 and let G^{rad} be the function in (2.7). Then,

$$G^{rad} = \sum_{j \in \{s, a\}} \int_{\mathcal{C}} \left[A_j^+(x, \xi; t) e^{i\beta_0 \alpha_+(x, z; \xi, \zeta; t)} + A_j^-(x, \xi; t) e^{i\beta_0 \alpha_-(x, z; \xi, \zeta; t)} \right] dt,$$

with

$$A_j^\pm(x, \xi; t) = \frac{1}{8\pi i} \cdot \frac{\Phi_j(x, \beta_0 \sin t) \Phi_j(\xi, \pm \beta_0 \sin t)}{\Phi_j(h, \beta_0 \sin t) \Phi_j(h, -\beta_0 \sin t)},$$

and

$$\alpha_\pm(x, z; \xi, \zeta; t) = ([x]_h \pm [\xi]_h) \sin t + |z - \zeta| \cos t,$$

$t \in \mathbb{C}$, and where Φ_j , $j \in \{s, a\}$, is given by (3.4). In particular, the following equivalent expression for G^{rad} will also be useful:

$$G^{\text{rad}} = \int_{\mathcal{C}} g(x, \xi; t) e^{i\beta_0([x]_h \sin t + |z - \zeta| \cos t)} dt, \quad (3.7)$$

where

$$g(x, \xi; t) = \sum_{j \in \{s, a\}} \left[A_j^+(x, \xi; t) e^{i[\xi]_h \sin t} + A_j^-(x, \xi; t) e^{-i[\xi]_h \sin t} \right]. \quad (3.8)$$

(Notice that g does not depend on x for $|x| \geq h$.)

Proof. We first take (2.7) and make the change of variable $\tau = \sqrt{\lambda - d^2}$ to obtain:

$$G^{\text{rad}} = \frac{1}{4\pi i} \sum_{j \in \{s, a\}} \int_{-\infty}^{+\infty} \frac{e^{i|z - \zeta| \sqrt{\beta_0^2 - \tau^2}}}{\sqrt{\beta_0^2 - \tau^2}} v_j(x, \tau^2 + d^2) v_j(\xi, \tau^2 + d^2) \sigma_j(\tau^2 + d^2) d\tau;$$

here, we also used the fact that all the relevant quantities subject to integration are even functions of τ . With the help of (3.5) and (3.6), and simple manipulations, we can infer that

$$G^{\text{rad}} = \frac{1}{8\pi i} \sum_{j \in \{s, a\}} \int_{-\infty}^{+\infty} \left\{ \frac{\Phi_j(x, \tau) \Phi_j(\xi, \tau)}{\Phi_j(h, \tau) \Phi_j(h, -\tau)} e^{i[\tau([x]_h + [\xi]_h) + |z - \zeta| \sqrt{\beta_0^2 - \tau^2}]} \right. \\ \left. + \frac{\Phi_j(x, \tau) \Phi_j(\xi, -\tau)}{\Phi_j(h, \tau) \Phi_j(h, -\tau)} e^{i[\tau([x]_h - [\xi]_h) + |z - \zeta| \sqrt{\beta_0^2 - \tau^2}]} \right\} d\tau.$$

The conclusion is then readily obtained by splitting the interval of integration up into the three intervals $(-\infty, -\beta_0)$, $[-\beta_0, \beta_0]$ and $(\beta_0, +\infty)$ and by subsequently making the change of variable $\tau = \beta_0 \sin t$, with $t \in \mathcal{C}$. \square

LEMMA 3.4. *For every ξ, ζ fixed, we have:*

$$\frac{\partial G^{\text{rad}}}{\partial x} = i\beta_0 \int_{\mathcal{C}} g(h \operatorname{sign} x, \xi; t) \sin t e^{i\beta_0([x]_h \sin t + |z - \zeta| \cos t)} dt, \quad (3.9a)$$

for $|x| \geq h$ and $z \neq \zeta$;

$$\frac{\partial G^{\text{rad}}}{\partial z} = i\beta_0 \operatorname{sign}(z - \zeta) \int_{\mathcal{C}} g(x, \xi; t) \cos t e^{i\beta_0([x]_h \sin t + |z - \zeta| \cos t)} dt, \quad (3.9b)$$

for $z \neq \zeta$.

In particular, on the set $(0, \zeta) + \partial\Omega_R$ given by (3.3), we have:

$$\frac{\partial G^{rad}}{\partial \nu} - i\beta_0 G^{rad} = i\beta_0 \int_{\mathcal{C}} g(x, \xi; t) [\cos t - 1] e^{i\beta_0 R \cos t} dt, \quad (3.10a)$$

for $z - \zeta = R$ and $|x| \leq h$, and

$$\frac{\partial G^{rad}}{\partial \nu} - i\beta_0 G^{rad} = i\beta_0 \int_{\mathcal{C}} g(h, \xi; t) [\cos(t - \vartheta) - 1] e^{i\beta_0 R \cos(t - \vartheta)} dt, \quad (3.10b)$$

where ν is the normal to $(0, \zeta) + \partial\Omega_R$ and we have set $[x]_h = R \sin \vartheta$ and $z - \zeta = R \cos \vartheta$ with $\vartheta \in [0, \pi/2)$ and $R > h$.

Formulas analogous to (3.10) hold for the remaining values of ϑ in $[-\pi, \pi)$.

Proof. Since $z \neq \zeta$ and $\text{Im}([x]_h \sin t + |z - \zeta| \cos t) \rightarrow +\infty$ as $t \rightarrow \infty$ on \mathcal{C} , the integrands in (3.9a) and (3.9b) vanish exponentially as $t \rightarrow \infty$ on \mathcal{C} , since g is bounded (see Lemma A.3). Thus, (3.9a) and (3.9b) follow from an application of Lebesgue's dominated convergence Theorem. \square

3.2. Uniform asymptotic estimates for $\frac{\partial G^{rad}}{\partial \nu} - i\beta_0 G^{rad}$. Aiming to estimate, as $R \rightarrow \infty$, the function $\frac{\partial G^{rad}}{\partial \nu} - i\beta_0 G^{rad}$ given by (3.10), we need to deform the contour \mathcal{C} to a more convenient one.

Without loss of generality we can assume that $\vartheta \in [0, \pi/2]$. We define the new contour \mathcal{C}_ϑ (see Fig. 3.3) as follows:

$$\mathcal{C}_\vartheta = \bigcup_{j=1}^5 \Gamma_j,$$

where

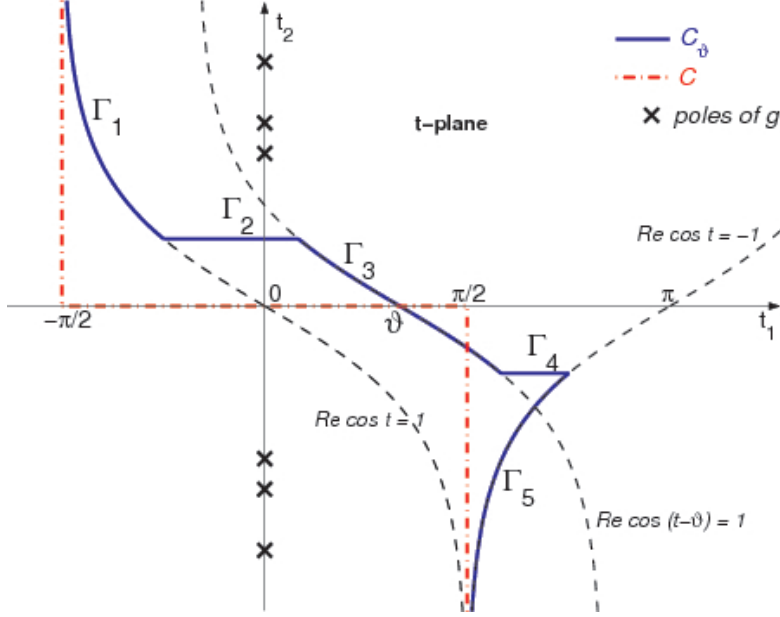
$$\delta_1 = \arccos \frac{2\beta_0}{\sqrt{4\beta_0^2 + d^2 - \gamma_M}}, \quad \delta_2 = \text{arcsinh} \frac{\sqrt{d^2 - \gamma_M}}{2\beta_0},$$

(notice that $\cos \delta_1 \cosh \delta_2 = 1$) and

$$\begin{aligned} \Gamma_1 &= \{t = t_1 + it_2 \in \mathbb{C} : \text{Re}(\cos t) = 1, \text{Im}(\cos t) \geq 0, -\frac{\pi}{2} < t_1 \leq -\delta_1, t_2 \geq \delta_2\}, \\ \Gamma_2 &= \{t \in \mathbb{C} : -\delta_1 \leq t_1 \leq -\delta_1 + \vartheta, t_2 = \delta_2\}, \\ \Gamma_3 &= \{t \in \mathbb{C} : \text{Re}[\cos(t - \vartheta)] = 1, \text{Im}[\cos(t - \vartheta)] \geq 0, |t_1 - \vartheta| \leq \delta_1, |t_2| \leq \delta_2\}, \\ \Gamma_4 &= \{t \in \mathbb{C} : \delta_1 + \vartheta \leq t_1 \leq \pi - \delta_1, t_2 = -\delta_2\}, \\ \Gamma_5 &= \{t \in \mathbb{C} : \text{Re}(\cos t) = -1, \text{Im}(\cos t) \geq 0, \frac{\pi}{2} < t_1 \leq \pi - \delta_1, t_2 \leq -\delta_2\}. \end{aligned}$$

This choice of \mathcal{C}_ϑ is suggested by the following three remarks:

- (i) $\mathcal{C} \cup \mathcal{C}_\vartheta$ does not contain in its interior the poles of g (which correspond to the guided part (2.6) of G);
- (ii) Γ_3 is part of the steepest descent path of $\cos(t - \vartheta)$;
- (iii) $\Gamma_1, \Gamma_2, \Gamma_4, \Gamma_5$ are chosen to complete the contour $\mathcal{C} \cup \mathcal{C}_\vartheta$ and to fulfill Lemma 3.5 below.

FIGURE 3.3. The contour \mathcal{C} .

By (i), it is clear that we can write

$$\frac{\partial G^{rad}}{\partial \nu} - i\beta_0 G^{rad} = i\beta_0 \int_{\mathcal{C}_0} g(x, \xi; t) [\cos t - 1] e^{i\beta_0 R \cos t} dt,$$

for $|x| \leq h$, and

$$\frac{\partial G^{rad}}{\partial \nu} - i\beta_0 G^{rad} = i\beta_0 \int_{\mathcal{C}_\vartheta} g(h, \xi; t) [\cos(t - \vartheta) - 1] e^{i\beta_0 R \cos(t - \vartheta)} dt,$$

for $x \geq h$.

LEMMA 3.5. *Let $(x, z) \in (0, \zeta) + \partial\Omega_R$. The following estimates hold for $R \rightarrow \infty$:*

$$\frac{\partial G^{rad}}{\partial \nu} - i\beta_0 G^{rad} = i\beta_0 \int_{\Gamma_3} g(x, \xi; t) [\cos t - 1] e^{i\beta_0 R \cos t} dt + \mathcal{O}(e^{-c\beta_0 R}), \quad (3.11a)$$

for $|x| \leq h$, and

$$\frac{\partial G^{rad}}{\partial \nu} - i\beta_0 G^{rad} = i\beta_0 \int_{\Gamma_3} g(h, \xi; t) [\cos(t - \vartheta) - 1] e^{i\beta_0 R \cos(t - \vartheta)} dt + \mathcal{O}(e^{-c\beta_0 R}), \quad (3.11b)$$

for $x \geq h$, $\vartheta \in [0, \pi/2]$, where

$$c = \sqrt{\frac{d^2 - \gamma_M}{4\beta_0^2 + d^2 - \gamma_M}} \cdot \min\left(1, \frac{\sqrt{d^2 - \gamma_M}}{2\beta_0}\right).$$

Proof. We shall prove only (3.11b) since (3.11a) follows analogously. We preliminarily observe that

$$\operatorname{Im} \cos(t - \vartheta) \geq c, \quad (3.12)$$

for $t \in \Gamma_1, \Gamma_2, \Gamma_4, \Gamma_5$ and $\vartheta \in [0, \pi/2]$. From (3.12), we easily obtain that

$$\left| \int_{\Gamma_j} g(h, \xi; t) [\cos(t - \vartheta) - 1] e^{i\beta_0 R \cos(t - \vartheta)} dt \right| \leq \frac{K\pi}{2} (\cosh \delta_2 + 1) e^{-c\beta_0 R}, \quad j = 2, 4,$$

where K is a bound for g (see Lemma A.3). Thus, it remains to prove that

$$\int_{\Gamma_j} g(h, \xi; t) [\cos(t - \vartheta) - 1] e^{i\beta_0 R \cos(t - \vartheta)} dt = \mathcal{O}(e^{-c\beta_0 R}), \quad j = 1, 5,$$

uniformly as $R \rightarrow \infty$, for $\vartheta \in [0, \pi/2]$. We carry out the details for $j = 1$, the case $j = 5$ is completely analogous. We first use Lemma A.2 to write that

$$\int_{\Gamma_1} g(h, \xi; t) [\cos(t - \vartheta) - 1] e^{i\beta_0 R \cos(t - \vartheta)} dt = J(R) + \mathcal{O}(e^{-c\beta_0 R}),$$

since (3.12) holds; here,

$$J(R) = i\beta_0 \int_{\Gamma_1} \left(1 + \frac{i}{2\beta_0 \sin t} \int_{\{\xi\}_h}^h p(y) dy \right) [\cos(t - \vartheta) - 1] e^{i\beta_0 [R \cos(t - \vartheta) + (h - \xi) \sin t]} dt,$$

with $p(y) = d^2 - q(y)$. Let $\psi(t) = R \cos(t - \vartheta) + (h - \xi) \sin t$ and $\delta = \delta_1 + i\delta_2$; an integration by parts yields

$$\begin{aligned} J(R) &= \frac{e^{i\beta_0 \psi(\delta)}}{\psi'(\delta)} \left(1 + \frac{i}{2\beta_0 \sin \delta} \int_{\{\xi\}_h}^h p(y) dy \right) [1 - \cos(\delta - \vartheta)] \\ &\quad + \int_{\Gamma_1} \frac{e^{i\beta_0 \psi(t)}}{\psi'(t)^2} \left\{ [\cos(t - \vartheta) - 1] \left[R - \frac{i\psi(t)}{2\beta_0 \sin t} \int_{\{\xi\}_h}^h p(y) dy \right] \right. \\ &\quad \left. + (h - \xi)(\sin t - \sin \vartheta) - \frac{i\psi'(t)(\cos t - \cos \vartheta)}{2\beta_0 \sin^2 t} \int_{\{\xi\}_h}^h p(y) dy \right\} dt. \end{aligned}$$

From (3.12) and since

$$\begin{aligned} \sinh t_2 &\leq |\cos t|, |\sin t| \leq \cosh t_2, \\ |\psi(t)| &\leq \beta_0(R + |h - \xi|) \cosh t_2, \quad |\psi'(t)| \geq \frac{1}{2}\beta_0 R \sinh t_2, \quad \text{for } R \geq 2|h - \xi| \coth \delta_2, \end{aligned}$$

for $t \in \Gamma_1$, we obtain that $J(R) = \mathcal{O}(e^{-c\beta_0 R})$, as $R \rightarrow \infty$. \square

THEOREM 3.6. *On $\partial\Omega_R$, we have*

$$\frac{\partial G^{rad}}{\partial \nu} - i\beta_0 G^{rad} = \mathcal{O}\left(R^{-\frac{3}{2}}\right), \quad (3.13)$$

uniformly as $R \rightarrow \infty$.

Proof. First, we estimate the left-hand side of (3.13) on the sets $(0, \zeta) + \partial\Omega_R$. By Lemma 3.5, we only need to estimate the first addendum in (3.11). We prove the estimate for (3.11b); the estimate for (3.11a) follows exactly in the same way.

Since Γ_3 is part of the steepest descent path, the steepest descent method (see [BO]) suggests to change the variables in the first addendum in (3.11b): by setting $\cos(t - \vartheta) = 1 + iy^2$, we obtain

$$\int_{\Gamma_3} g(h, \xi; t) [\cos(t - \vartheta) - 1] e^{i\beta_0 R \cos(t - \vartheta)} dt = -4ie^{i\beta_0 R} \int_0^{y_0} y^2 e^{-\beta_0 R y^2} \frac{g(h, \xi; t(y))}{\sqrt{y^2 - 2i}} dy,$$

with $y_0 = (\sin \delta_1 \sinh \delta_2)^{\frac{1}{2}}$. Thanks to Lemma A.3,

$$\left| \int_{\Gamma_3} g(h, \xi; t) [\cos(t - \vartheta) - 1] e^{i\beta_0 R \cos(t - \vartheta)} dt \right| \leq 2\sqrt{2}K \int_0^{y_0} y^2 e^{-\beta_0 R y^2} dy \leq K \sqrt{\frac{\pi}{2\beta_0^3}} R^{-\frac{3}{2}},$$

where K is a bound of g . Therefore, (3.11b) implies that

$$\left| \frac{\partial G^{rad}}{\partial \nu} - i\beta_0 G^{rad} \right| \leq K \sqrt{\frac{\pi}{2\beta_0}} R^{-\frac{3}{2}},$$

on the sets $(0, \zeta) + \partial\Omega_R$.

By using exactly the same argument as before, we can prove that the derivatives of G^{rad} are $\mathcal{O}(R^{-\frac{1}{2}})$ on the sets $(0, \zeta) + \partial\Omega_R$, uniformly as $R \rightarrow \infty$; we reach the conclusion (3.13) by observing that $\nu_{\partial\Omega_R} - \nu_{(0, \zeta) + \partial\Omega_R} = \mathcal{O}(R^{-1})$, as $R \rightarrow \infty$. \square

3.3. Proof of Theorem 3.1. Since $f \in L^2(\mathbb{R}^2)$ and f has compact support, from Corollary 5.1 in [CM] we have that $u \in L^{2,s}(\mathbb{R}^2)$, $s < -1$.

Thus, it remains to prove that (2.1) satisfies (2.19). In order to do it, we shall check the following facts:

- (i) if u is given by (2.1) and u_l , $l = 1, \dots, M$, is computed via (2.16), the remainder part u_0 of u , given by (2.18), equals the function u^{rad} in (2.9);
- (ii) u satisfies (3.2).

We preliminarily notice that

$$\int_0^{+\infty} \int_{\partial\Omega_R} \left| \frac{\partial u_0}{\partial \nu} - i\beta_0 u_0 \right|^2 d\ell dR < +\infty$$

is easily verified thanks to Theorem 3.6.

The following property of orthogonality is useful to check (i).

LEMMA 3.7. *Let $e(x, \gamma_l)$, $l = 1, \dots, M$, and $v_j(x, \lambda)$, $j \in \{s, a\}$, be the solutions of (2.4) given by (2.11) and (2.5), respectively, with $\lambda > 0$. If $\lambda \neq \gamma_l$, then*

$$\int_{-\infty}^{+\infty} e(x, \gamma_l) v_j(x, \lambda) dx = 0.$$

Proof. We multiply the following equations

$$\begin{aligned} e''(x, \gamma_l) + [\gamma_l - q(x)]e(x, \gamma_l) &= 0, \\ v''(x, \lambda) + [\lambda - q(x)]v(x, \lambda) &= 0, \end{aligned}$$

by $v(x, \lambda)$ and $e(x, \gamma_l)$, respectively, and integrate in x over an interval (a, b) . An integration by parts gives:

$$\begin{aligned} (\gamma_l - \lambda) \int_a^b e(x, \gamma_l) v(x, \lambda) dx &= \int_a^b [e(x, \gamma_l) v''(x, \lambda) - e''(x, \gamma_l) v(x, \lambda)] dx \\ &= [e(x, \gamma_l) v'(x, \lambda) - e'(x, \gamma_l) v(x, \lambda)]_a^b. \end{aligned}$$

The conclusion follows by observing that $e(x, \gamma_l)$ and its first derivative vanish exponentially as $|x| \rightarrow \infty$, while $v(x, \lambda)$ and $v'(x, \lambda)$ are bounded. \square

Now, by (2.1), (2.16) and Lemma 3.7, we have that

$$u_l(x, z) = \int_{\mathbb{R}^2} G_l^g(x, z; \xi, \zeta) f(\xi, \zeta) d\xi d\zeta, \quad l = 1, \dots, M,$$

with G_l^g given by (2.12) and thus $u_0 = u^{rad}$.

To complete the proof it remains to check (ii) for $l = 1, \dots, M$. When z is large enough, we have

$$\frac{\partial u_l}{\partial \nu} - i\beta_l u_l = 0, \quad l = 1, \dots, M,$$

on $\partial Q_R \cap \{(x, z) : |z| = R\}$, since $\frac{\partial}{\partial \nu} = \pm \frac{\partial}{\partial z}$. Thanks to (2.10), we easily find that

$$\left| \frac{\partial u_l}{\partial \nu} - i\beta_l u_l \right| = \mathcal{O}\left(e^{-R\sqrt{d^2 - \gamma_l}}\right),$$

as $R \rightarrow \infty$ on $\partial Q_R \cap \{(x, z) : |x| = R\}$ and thus we obtain that

$$\int_0^{+\infty} \int_{\partial Q_R} \left| \frac{\partial u_l}{\partial \nu} - i\beta_l u_l \right|^2 d\ell dR < +\infty, \quad l = 1, \dots, M,$$

which completes the proof.

Appendix A. Asymptotic Lemmas. In what follows, $BV(\mathbb{R})$ denotes the space of functions with bounded variation.

LEMMA A.1. *Let T be a non-negative number, $q \in BV(\mathbb{R})$ and*

$$p(x) = d^2 - q(x), \quad x \in \mathbb{R}.$$

Then, the following asymptotic estimates for the functions Φ_s and Φ_a given by (3.4) hold uniformly as $|\tau| \rightarrow +\infty$, for $x \in \mathbb{R}$ and $|\operatorname{Im} \tau| \leq T$:

$$\Phi_s(x, \tau) = \left[1 + \frac{i}{2\tau} \int_0^{\{x\}_h} p(y) dy \right] e^{i\tau\{x\}_h} + \mathcal{O}\left(\frac{1}{|\tau|^2}\right), \quad (\text{A.1})$$

$$\Phi_a(x, \tau) = \frac{\sqrt{\tau^2 + d^2}}{i\tau} \left[1 + \frac{i}{2\tau} \int_0^{\{x\}_h} p(y) dy \right] e^{i\tau\{x\}_h} + \mathcal{O}\left(\frac{1}{|\tau|^2}\right). \quad (\text{A.2})$$

Proof. (i) First, we prove an estimate for ϕ_j , $j \in \{s, a\}$. From (2.4), we know that ϕ_j satisfies

$$\phi_j''(y, \lambda) + [\tau^2 + p(y)]\phi_j(y, \lambda) = 0, \quad y \in [-h, h].$$

We multiply the above equation by $\sin \tau(x - y)$, integrate by parts twice and obtain the following integral equation:

$$\phi_j(x, \lambda) = \frac{\phi_j'(0, \lambda)}{\tau} \sin \tau x + \phi_j(0, \lambda) \cos \tau x - \frac{1}{\tau} \int_0^x p(y) \sin(\tau(x - y)) \phi_j(y, \lambda) dy. \quad (\text{A.3})$$

We set $\eta_j(x, \lambda) = \sup_{s \in [0, x]} |\phi_j(s, \lambda)|$. Since $|\sin \tau x|, |\cos \tau x| \leq \cosh \tau_2 x$ ($\tau_2 = \operatorname{Im} \tau$), from the above equation we have that

$$\eta_j(x, \lambda) \leq \left[\frac{|\phi_j'(0, \lambda)|}{|\tau|} + |\phi_j(0, \lambda)| \right] \cosh \tau_2 x + \frac{1}{|\tau|} \int_0^x p(y) \cosh \tau_2(x - y) \eta_j(y, \lambda) dy,$$

and, by Gronwall's Lemma (see [SC]), we get

$$\begin{aligned} \eta_j(x, \lambda) &\leq \left[\frac{|\phi_j'(0, \lambda)|}{|\tau|} + |\phi_j(0, \lambda)| \right] e^{\frac{1}{|\tau|} \int_0^x p(y) \cosh \tau_2(x-y) dy} \times \\ &\quad \times \left\{ 1 + \tau_2 \int_0^x e^{-\int_0^s p(y) \cosh \tau_2(x-y) dy} \sinh \tau_2 s ds \right\}. \end{aligned}$$

Since $0 \leq p(y) \leq d^2$, we have that

$$\eta_j(x, \lambda) \leq \left[\frac{|\phi_j'(0, \lambda)|}{|\tau|} + |\phi_j(0, \lambda)| \right] \cosh \tau_2 x \exp \left\{ \frac{d^2 \sinh \tau_2 x}{|\tau| \tau_2} \right\}.$$

If we assume $|\tau| \geq d$ and $x \in [-h, h]$, we finally get

$$|\phi_j(x, \lambda)| \leq \sqrt{2} \cosh \tau_2 h \exp \left\{ \frac{d \sinh Th}{T} \right\}, \quad j \in \{s, a\}. \quad (\text{A.4})$$

(ii) Now we prove (A.1) and (A.2). Let $q \in C^1(\mathbb{R})$. From (3.4), by straightforward manipulations we have:

$$\Phi_j'(x, \tau) - i\tau \Phi_j(x, \tau) = \frac{i}{\tau} p(x) \phi_j(x, \lambda); \quad (\text{A.5})$$

by multiplying the above equation by $e^{-i\tau x}$, integrating by parts twice and observing that

$$\begin{aligned} 2 \int_0^x e^{-i\tau y} p(y) \phi_j(y, \lambda) dy &= \int_0^x e^{-i\tau y} p(y) \Phi_j(y, \lambda) dy \\ &+ \frac{1}{i\tau} \int_0^x e^{-i\tau y} p'(y) \phi_j(y, \lambda) dy - \left[\frac{e^{-i\tau y}}{i\tau} p(y) \phi_j(y, \lambda) \right]_{y=0}^{y=x}, \end{aligned}$$

it follows that Φ_j satisfies

$$\begin{aligned} \Phi_j(x, \tau) e^{-i\tau x} &= \Phi_j(0, \tau) + \frac{i}{2\tau} \int_0^x e^{-i\tau y} p(y) \Phi_j(y, \lambda) dy \\ &+ \frac{1}{2\tau^2} \left\{ \int_0^x e^{-i\tau y} p'(y) \phi_j(y, \lambda) dy - e^{-i\tau x} p(x) \phi_j(x, \lambda) + p(0) \phi_j(0, \lambda) \right\}. \quad (\text{A.6}) \end{aligned}$$

By setting $M_j(x, \tau) = \sup_{s \in [0, x]} |\Phi_j(s, \lambda) e^{-i\tau s}|$ and from (A.4), we get

$$\begin{aligned} M_j(x, \tau) &\leq |\Phi_j(0, \tau)| + \frac{1}{2|\tau|} \int_0^x p(y) M_j(y, \tau) dy \\ &+ \frac{1}{2|\tau|^2} \left\{ C \int_0^x e^{\tau_2 y} |p'(y)| dy + Cd^2 e^{\tau_2 x} + p(0) |\phi_j(0, \lambda)| \right\}, \end{aligned}$$

for $|\tau| \geq d$, where C is the right-hand side of (A.4). Thus, Gronwall's Lemma yields the following estimate for M_j :

$$\begin{aligned} M_j(x, \tau) &\leq \left[|\Phi_j(0, \tau)| + \frac{Cd^2}{2|\tau|^2} + p(0) |\phi_j(0, \lambda)| \right] \exp \left\{ \frac{1}{2|\tau|} \int_0^x p(y) dy \right\} \\ &+ \frac{C}{2|\tau|^2} \int_0^x \exp \left\{ \frac{1}{2|\tau|} \int_s^x p(y) dy \right\} e^{\tau_2 s} [|p'(s)| + \tau_2 d^2] ds; \end{aligned}$$

since $\Phi_s(0, \tau) = 1$, $\Phi_a(0, \tau) = \frac{\sqrt{\tau^2 + d^2}}{i\tau}$ and $0 \leq p(x) \leq d^2$, we have

$$M_j(x, \tau) \leq e^{\frac{dh}{2}} \left\{ 1 + \sqrt{2} + \frac{C}{2d^2} e^{Tx} [d^2 + |q|_{BV}] \right\}, \quad (\text{A.7})$$

for $|\tau| \geq d$. By a standard approximation argument we can infer that (A.7) holds for every $q \in BV(\mathbb{R})$. By (A.6), (A.4) and (A.7), we get that

$$\Phi_j(x, \tau) = \Phi_j(0, \tau) e^{i\tau x} + \mathcal{O}\left(\frac{1}{|\tau|}\right).$$

Again, from (A.6) and the above asymptotic formula, we obtain (A.1) and (A.2). \square

In Lemmas A.2 and A.3, we will use the following inequality:

$$|\operatorname{Im} \sin t| \leq \max \left\{ 1, \frac{\sqrt{d^2 - \gamma_M}}{2\beta_0} \right\}, \quad t \in \mathcal{C}_\vartheta, \quad \vartheta \in [0, \pi/2]. \quad (\text{A.8})$$

LEMMA A.2. *Let g be defined by (3.8). Then, the following asymptotic expansion*

$$g(x, \xi; t) = \frac{1}{4\pi i} e^{i\beta_0(\{x\}_h - \xi) \sin t} \left[1 + \frac{i}{2\beta_0 \sin t} \int_{\{\xi\}_h}^{\{x\}_h} p(y) dy \right] + \mathcal{O}\left(\frac{1}{|\sin t|^2}\right) \quad (\text{A.9})$$

holds uniformly as $t \rightarrow \infty$ on \mathcal{C}_ϑ for $\vartheta \in [0, \pi/2]$, $x \in \mathbb{R}$ and ξ bounded.

Proof. The proof is a straightforward consequence of Lemma A.1 and (A.8), and hence is omitted. \square

LEMMA A.3. *Let g be given by (3.8). Then g is a bounded function of $x, \xi \in \mathbb{R}$, if ξ is bounded, and $t \in \mathcal{C}_\vartheta$, $\vartheta \in [0, \pi/2]$.*

Proof. (i) First, we prove an estimate for $\phi_j(x, \tau^2 + d^2)$ for $|\tau| \leq d$, $|\operatorname{Im} \tau| \leq T$ and $|x| \leq h$. By setting $\lambda = \tau^2 + d^2$ and $\eta_j(x, \lambda) = \sup_{s \in [0, x]} |\phi_j(s, \lambda)|$ as before, from

(A.3) and since $|\frac{\sin \tau x}{\tau x}|$ is bounded by the constant $B = \sqrt{\cosh^2(Th) + \frac{\sinh^2(Th)}{(Th)^2}}$, we have

$$\begin{aligned} \eta_s(x, \lambda) &\leq \cosh(Tx) + B \int_0^{|x|} p(y) |x - y| \eta_s(y, \lambda) dy, \\ \eta_a(x, \lambda) &\leq B \left\{ \sqrt{2}d|x| + \int_0^{|x|} p(y) |x - y| \eta_a(y, \lambda) dy \right\}; \end{aligned}$$

Gronwall's Lemma yields

$$|\phi_j(x, \lambda)| \leq \min \left\{ \cosh(Th), \sqrt{2}dhB \right\} \exp \left(B \frac{d^2 h^2}{2} \right), \quad (\text{A.10})$$

for $|x| \leq h$, $|\tau| \leq d$ and $|\operatorname{Im} \tau| \leq T$.

(ii) To complete the proof, we notice that from (A.5) it follows that

$$\left| \frac{\Phi_j(x, \tau)}{\Phi_j(h, \tau)} \right| \leq \frac{de^{Th}}{|\tau \Phi_j(h, \tau)|} \left(\sqrt{2} + dhe^{Th} \sup_{x \in [-h, h]} |\phi_j(x, \lambda)| \right),$$

and since $\tau \Phi_j(h, \tau) \neq 0$ far from the poles of g , we have that $\frac{\Phi_j(x, \tau)}{\Phi_j(h, \tau)}$ is bounded for $\tau = \beta_0 \sin t$, $t \in \mathcal{C}_\vartheta$, $\vartheta \in [0, \pi/2]$ and for $|\tau| \leq d$. Thus, the assertion of the lemma follows from (A.10) and Lemma A.2. \square

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